PRIFYSGOL ABERYSTWYTH UNIVERSITY

ENTRANCE EXAMINATION
MAY 2021

FURTHER MATHEMATICS

Time allowed: 1 hour 30 minutes

- All answers (including any diagrams, graphs or sketches) should be written on paper, and scanned into a **single** PDF file. Graph paper is not required.
- Answer **all** questions in Section A and **two** questions from Section B.
- Candidates are permitted to use calculators, provided they comply with A level examining board regulations. They must be made available on request for inspection by invigilators, who are authorised to remove any suspect calculators.
- Statistical tables are not required.

Information

- 2-D rotations and reflections are represented by matrices as follows.

Anticlockwise rotation through angle ϕ about the origin : $\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$

Reflection in the line $y = (\tan\phi)\,x$: $\begin{pmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{pmatrix}$

Section A

1. Simplify the following expressions as far as possible, showing your workings.

(a) $\frac{2}{i}\,\frac{2i+3}{1-i} + i$ [3 marks]

(b) $(3\mathbf{i} + 4\mathbf{k}) \cdot (-\mathbf{i} + 5\mathbf{j} + \mathbf{k})$ [2 marks]

(c) $\begin{pmatrix} 3 & 5 \\ 2 & 4 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ [5 marks]

2. The complex number z satisfying $3|z - 1 + i| = 2|z + i|$ is represented by the point $P(x, y)$ in an Argand diagram. Show that the locus of P is a circle, and find its radius and centre. [8 marks]

3. Consider the cubic equation $2x^3 - 3x^2 + 6x + 65 = 0$.

(a) Show that $2 - 3i$ is a root of the equation. [3 marks]

(b) Find the other two roots of the equation, explaining your method. [4 marks]

4. (a) (i) Differentiate $x^{-1} \sin x$ and $x^{-1} \cos x$ with respect to x. (ii) Hence calculate $\int_{\pi/2}^{\pi} \left(\frac{3+x}{x^2} \cos x - \frac{1-3x}{x^2} \sin x \right) dx$, showing all workings. [7 marks]

(b) Show that $\int \frac{dx}{1-x^2} = \frac{1}{2} \ln \frac{1+x}{1-x} + C$ (when $-1 < x < 1$). [4 marks]

5. Prove by mathematical induction that

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^n = \frac{1}{2} \begin{pmatrix} 3^n + 1 & 3^n - 1 \\ 3^n - 1 & 3^n + 1 \end{pmatrix},$$

for all positive integers n. [7 marks]

6. The plane Π contains the origin and is perpendicular to the vector $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$. The line L passes through the points $A(1, 1, 1)$ and $B(1, -1, -1)$.

(a) Write down the equation of the plane Π in Cartesian form. [1 mark]

(b) Show that the point $(1, 1, -1)$ lies in Π. [1 mark]

(c) Find (i) the vector $\mathbf{AB}$, and (ii) the vector equation of the line L. [3 marks]

(d) Find the coordinates of the point of intersection of L and Π. [3 marks]

7. Consider the following transformations in the plane:

R_1 is the anticlockwise rotation by $\frac{\pi}{4}$ radians about the origin.

R_2 is the reflection in the line $y + \sqrt{3}x = 0$.

(a) Find the 2×2-matrices representing R_1 and R_2. Simplify the answer using exact values of sines and cosines. [5 marks]

(b) The transformation T consists of R_1 followed by R_2 followed by the inverse of R_1. Find the matrix representing T. [4 marks]

Section B

8. Consider the function $f(x) = 2a(3-2x)x^2 + (3x-4)x^3$, where a is a constant such that $0 < a < 1$.

(a) Show that $(0,0)$, $(a, a^3(2-a))$ and $(1, 2a-1)$ are (all) the stationary points of the curve $y = f(x)$. [4 marks]

(b) Determine the nature of each stationary point. Make sure to indicate where you use the condition $0 < a < 1$ in your solution. [4 marks]

(c) Write down (in terms of a) the greatest value of the function $f(x)$ in the interval $0 \le x \le 1$. Justify your answer. [2 marks]

(d) Assuming $\frac{1}{2} \le a < 1$, explain why $f(x) \ge 0$ for all $0 \le x \le 1$. [1 mark]

(e) Assume now $0 < a < \frac{1}{2}$. How many roots does the equation $f(x) = 0$ have in the interval $0 \le x \le 1$? Justify your answer carefully. [5 marks]

(f) Consider the following statements A and B, where $0 \le x \le 1$ and $0 < a < 1$:

A If $\frac{1}{2} \le a < 1$ and $f(x) = 0$, then $x = 0$.

B If $0 < x \le 1$ and $f(x) = 0$, then $0 < a < \frac{1}{2}$.

In both cases, identify whether the statement is true or false. Justify your answer by giving a proof (if true), or a counterexample (if false). [4 marks]

9. A ball of mass m kg is dropped from the roof of a tall building (with zero initial velocity). As the ball falls, the air exerts on it a resistive force with magnitude kv^2 N, where k is a constant and v ms^{-1} is the velocity of the ball at time t.

(a) In a diagram, show the direction of motion and all the forces acting on the ball. [2 marks]

(b) Write down the equation of motion of the ball in terms of the velocity v, the acceleration a, and the constants g, m, k. [2 marks]

(c) Explain briefly why the acceleration decreases as the ball falls. [2 marks]

(d) The limit of zero acceleration defines the *terminal speed* v_T. Using the equation of motion, show that $v_T = \sqrt{mg/k}$. [2 marks]

(e) Show that the equation of motion can be written as

$$\frac{dv}{dt} = g\left(1 - \left(v/v_T\right)^2\right).$$

[3 marks]

(f) Using the integral formula in question 4(b), solve the differential equation in (e) to find an expression for $v = v(t)$ as a function of time t. [7 marks]

(g) Using the expression you found in (f), find the limit of $v(t)$ as $t \to \infty$, and interpret the result in this context. [2 marks]

10. A random variable, X, has the following probability distribution:

x	$\mathrm{P}(X = x)$
0	p
1	$2p$
2	q
3	$2q$
4	0.15
5	0.1

In addition, $\mathrm{E}(X) = 2.5$.

(a) Write down two equations that p and q must satisfy, and hence find the values of p and q. [6 marks]

(b) Find $\mathrm{Var}(X)$. [3 marks]

Adrian and Jemima play a game. A value, X, is randomly selected from the distribution above. Adrian's score is given by the random variable $A = X$, and Jemima's score is given by the random variable $J = 5 - X$.

(c) Find $\mathrm{E}(J)$. [2 marks]

Adrian and Jemima work out their scores and whoever has the higher score is the winner.

(d) What is the probability that Adrian wins? [3 marks]

(e) What is the probability that Jemima wins? [1 mark]

(f) A third player, Caroline, joins the game and her score is the random variable $C = X^2 + 1$.

(i) Explain clearly why Caroline will always beat Adrian. [2 marks]

(ii) Calculate the probability that Jemima beats Caroline. [3 marks]